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Using Force to Crack Some Geometry Chestnuts

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Abstract

The symbolic mechanics software "Mechanical Expressions" is capable of expressing static problems where force elements such as springs, masses and actuators interact with geometric elements such as points, lines and circles. It is able to derive symbolic expressions for resultant forces in constraints. Finding an equilibrium for the system involves finding constraint values such that the resultant forces are zero. This can be done in a CAS. In this paper we show how to apply this technique to solve some classic geometry optimization problems.

1. Introduction

In his book "The Mathematical Mechanic" [1], Mark Levi uses physical thought experiments to prove mathematical results. In this paper, we apply the symbolic mechanics program "Mechanical Expressions"[MechEpressions] to solve a number of classical geometry problems in the style of Levi.

The archetype of Levi's approach is the following physical proof of the location of the Fermat Toricelli Point. Given a triangle (none of whose angles exceeds 120 degrees), we are to find the location which minimizes the sum of the distances to the vertices.

We imagine boring holes in a table at the vertices of the triangle. We now attach three equal masses to three pieces of string, thread the strings through the holes and tie them together in a knot. At this point we have the masses dangling below the table and the knot above the table. Our physical thought experiment should settle at a position of minimal potential energy. As the masses are equal, the potential energy of the system is proportional to the sum of the heights of the masses, hence when PE is minimized the amount of string below the table is maximized, and hence the amount of string above the table is minimized. Hence when the potential energy is minimized, the knot will lie at the Fermat Toricelli point.

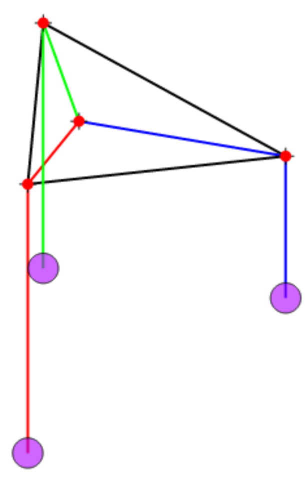


Figure 1: A simulation of the mechanical model for finding the Fermat Toricelli Point

To determine the location of the point, we consider the forces in the strings. Each string has the same tension (equivalent to the gravitational force on a single mass). At equilibrium the three force vectors, equal in length must add to 0, and hence must form an equilateral triangle. Hence the Fermat Toricelli point subtends 120 degrees to each side of the triangle. Figure 1 shows a mechanics simulation of this physical model (turned on its side). An interactive version of the simulation is online [S1].

There are two places in the above proof where cleverness is needed. First we need to be clever enough to construct a physical model whose equilibrium position corresponds to the solution of the geometric problem. Secondly, we need to extract geometric information from a consideration of the equilibrium position. In this case the second piece of cleverness was to consider the fact that the forces in the system must balance at equilibrium, and being equal in magnitude must form an equilateral triangle.

In this paper, we examine a number of geometrical optimization problems and create physical models in Mechanical Expressions, whose equilibrium positions correspond to the solution of the geometric problem. We then use the symbolic mechanics capabilities of Mechanical Expressions to determine geometric conditions for equilibrium directly. A CAS is used to solve these equilibrium equations, whose results provide explicit, albeit algebraic, locations for the solution. When the results are in turn analysed in Mechanical Expressions, a more geometrical characterization of the solution may be obtained.

2. Mechanical Expressions

Mechanical Expressions is built on top of the constraint based symbolic geometry program Geometry Expressions [GeomExpressions]. Geometry Expressions provides a mapping from a set of constraints to a Cartesian geometry. Some constraints may be regarded as variables in a model, others as parameters. All can be symbolic.

Mechanical Expressions allows the user to define mechanical quantities: masses, force elements, velocities and acceleration and derives a Lagrangian model of the mechanics [2]. This model is used by the system to determine resultant forces, velocities, accelerations and equations of motion.

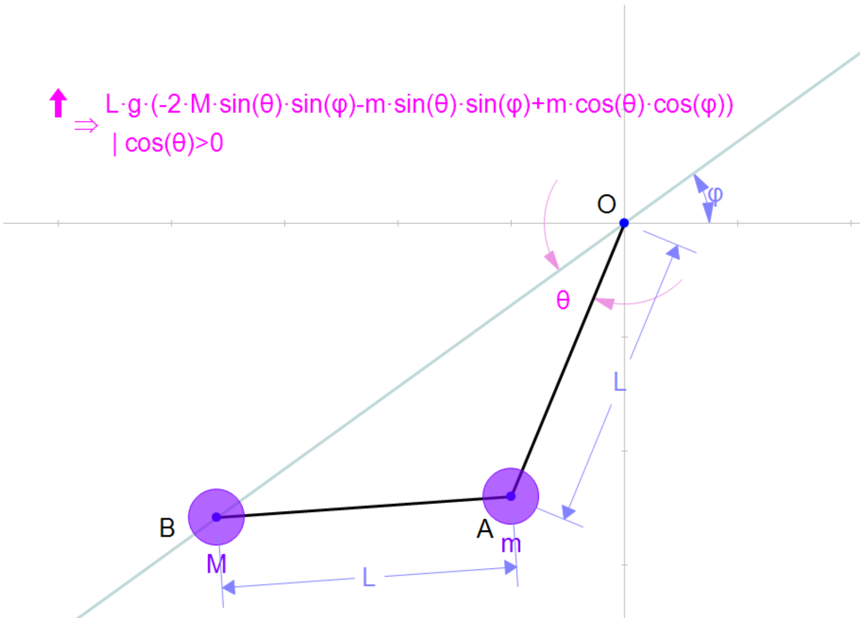


Figure 2: A Mechanical Expressions model of a double pendulum where the second mass is constrained to lie on a particular slope.

In a Mechanical Expressions model, it is the constraints which carry the reactions to forces applied to the model. For example, figure 2 shows a Mechanical Expressions model of a double pendulum, where the bob of the second pendulum is constrained to lie on a specific sloping line. The angle of the sloping line is φ, considered as a constant. The lengths of the pendulum arms are both constrained to be L, also considered to be constant. The angle between the first arm and the sloping line is defined to be θ. Bobs of mass m and M are positioned at the ends of the two pendulum arms. The constrained double pendulum has one degree of freedom, which corresponds to the angle constraint θ in the figure. The reaction force (actually a torque) in this constraint due to the gravitational forces on the two masses may be computed by Mechanical Expressions, and is shown in the figure.
For the model to be in static equilibrium, this reaction force must be zero. Hence the equilibrium position may be found by solving this equation (figure 3). An interactive illustration of this solution can be found online [S2].

```
> solve((((sin(phi)*sin(theta)*M*(-2))+(sin(phi)*sin(theta)*m* (-1))+(cos(phi)*cos(theta)*m))*g*L),theta);
```

$$\arctan\left(\frac{m}{\tan(\phi)\,(2\,M+m)}\right)$$

Figure 3: Maple solution for equilibrium position of constrained double pendulum

3. A Ladder Problem

A classic problem is to determine the size of the largest ladder which can fit round the corner between two corridors of unequal width [Todd 2009, Kalman 2007]. Levi [1] observes that the largest ladder which fits round the corner must at some point in its motion simultaneously touch the outside walls of the corridors and the corner point (figure 4). Hence the problem of finding the largest ladder which fits round the corner is equivalent to finding the length of the shortest line segment AB in figure 1 which passes through C.

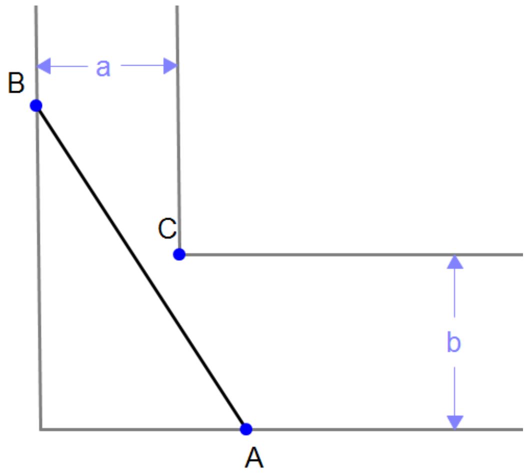
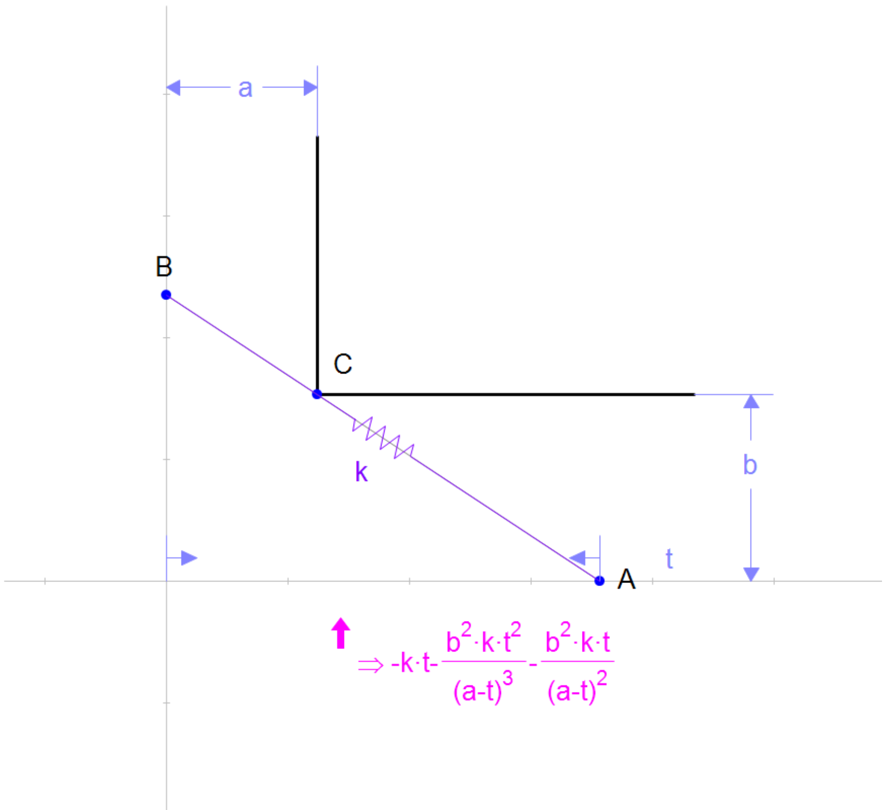


Figure 4: AB is a ladder which fits round the corner between corridors of width a and b. If it is the largest such ladder, it must touch C at some point in the motion.

A physical thought experiment whose equilibrium position corresponds to the solution of this problem is described in [1]. Imagine a spring-loaded telescoping bar, which passes through a hinged sleeve mounted at point C. The ends of the bar are constrained to lie on the opposite walls of the corridor. An advantage to thought experiments is that we can use idealized elements. It is convenient to postulate a spring with zero natural length. As the potential energy in a spring is proportional to the square of its extension (the difference between its length and its natural length), a zero natural length spring has potential energy proportional to its length. For this problem, use of such a spring (rather than a fixed force actuator, or a spring with finite natural length) leads to simpler mathematics.

Figure 5 shows a Mechanical Expressions model of this thought experiment. Point C is constrained to lie on line AB. B is constrained to lie on the y axis, and point A is constrained to lie at parametric location (x-coordinate) t on the x axis. A spring is added with end points A and B. The spring has stiffness k and natural length 0. The resultant force in the parameter t is computed by the software. [S3] is an interactive simulation of this mechanism.




```
> solve(((t*k*(-1))+((((a*(-1))+t))^((-
3))*(t)^(2)*k*(b)^(2))+((((a*(-1))+t))^((-2))*t*k*(b)^(2)*(-1))),t);
```

$$0,\ (a\,b^2)^{(1/3)} + a,\ -\frac{1}{2}(a\,b^2)^{(1/3)} + \frac{1}{2}I\sqrt{3}\,(a\,b^2)^{(1/3)} + a,\ -\frac{1}{2}(a\,b^2)^{(1/3)} - \frac{1}{2}I\sqrt{3}\,(a\,b^2)^{(1/3)} + a$$

Figure 5: (a) Mechanical Expressions model for the ladder problem. The reaction force in the parameter t is computed. (b) Maple solution for equilibrium.

To find an equilibrium position we need to solve for t such that the reaction force is zero. Figure 5b shows the Maple solution of the equation. One solution is zero and two are complex. There is one real solution. This can be copied back into the parametric location parameter in Mechanical Expressions (figure 6).

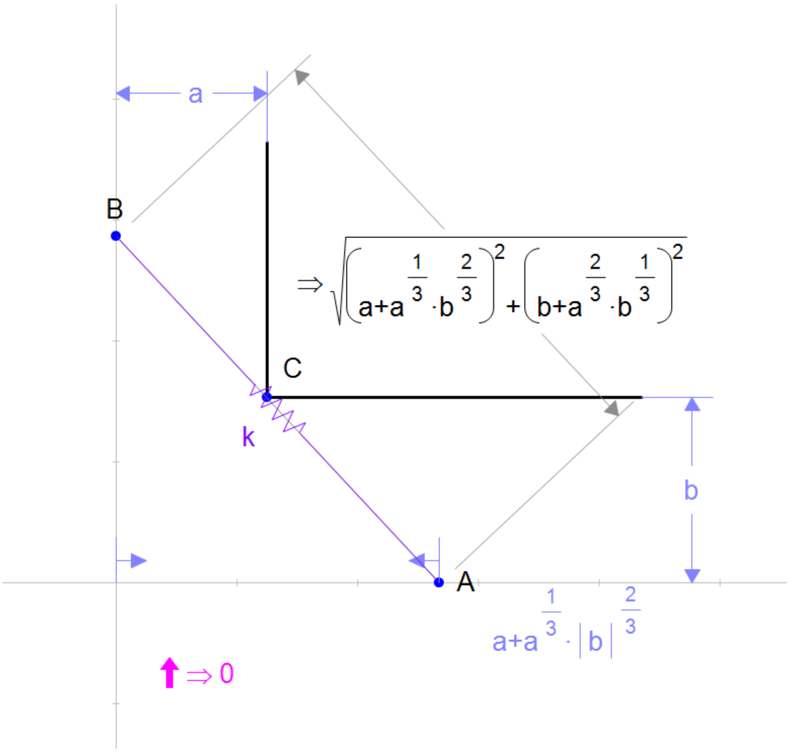


Figure 6: The Maple solution is used for the parametric location of A, and the critical length computed.

This can be simplified to yield the classic solution:

$$\left(a^{2/3}+b^{2/3}\right)^{\frac{3}{2}}$$

4. Regiomontanus Problem

Regiomontanus problem, described in [5] is to find the best place on earth to observe the rings of Saturn. A special case of this problem (the flat-earth version) can be phrased as the best place to watch a drive in movie, or the best place to stand in an art-gallery, or even as the best place to take a conversion kick in rugby [6].
Levi [1] uses the drive in movie setting assuming the screen has height a and its bottom edge is height h above your head. Where should you park in order to have the best view of the screen? Figure 7 shows a Mechanical Expressions model for solving this problem. AB represents the screen, which is located on the y axis. A is constrained to be distance h above the x axis, while AB is constrained to have length a. Point C is constrained to lie at parametric location t on the x-axis and lines CA and CB drawn. Our problem is to determine a value of t which maximizes the angle ACB. In Mechanical Expressions, we can add a rotational actuator between the two lines. We define the actuator to supply a constant torque between the lines. The potential energy in such an actuator is equal to the torque times the angle, and hence will be in equilibrium when the angle is at an extreme. [S4] is an online version of this model.

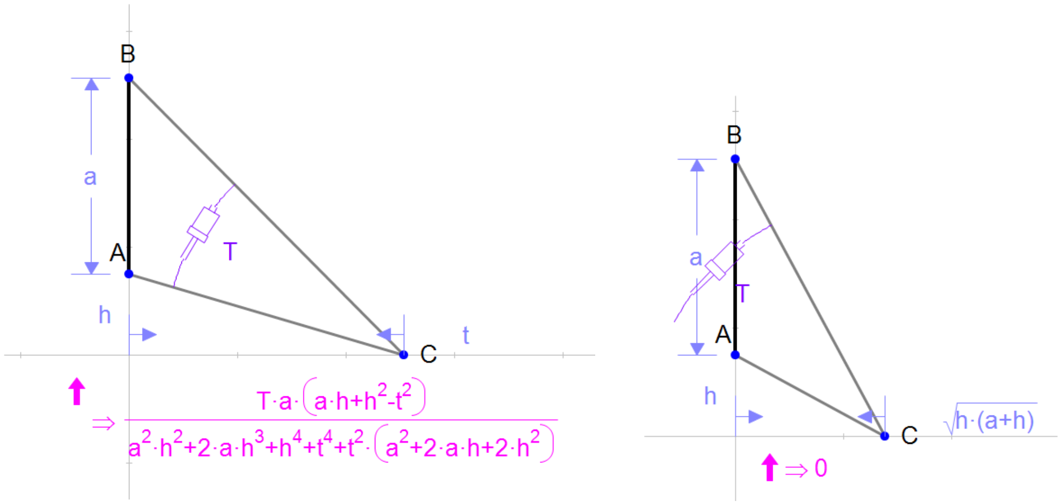
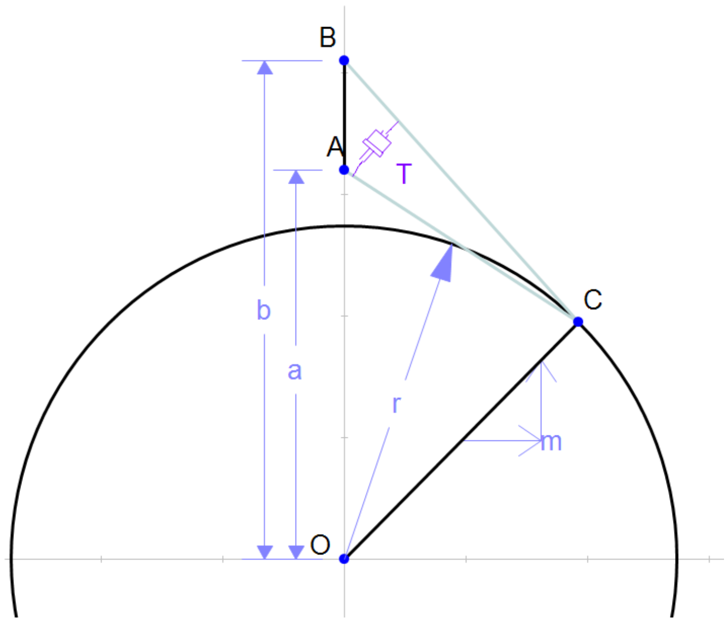


Figure 7: (a) Mechanical Expressions Model for the "drive-in movie" problem. (b) Equilibrium position.

Figure 7a shows the reaction in the parameter t due to this applied torque. Inspection of the numerator leads to an equilibrium solution when $t = \sqrt{h(a+h)}$. Figure 7b shows the solution.



$$\left(\left(-r\,a-r\,b-r\,m^2\,a-r\,m^2\,b+\left(m\,b\,a+r^2\,m\right)\sqrt{1+m^2}\right)(a-b)\,r\,T\right)\Big/\left(\left(b^2\,a^2+m^2\,b^2\,a^2+r^2\,a^2+r^2\,b^2+r^2\,m^2\,a^2+4\,r^2\,m^2\,b\,a+r^2\,m^2\,b^2+r^4+r^4\,m^2+\left(-2\,r\,m\,b\,a^2-2\,r\,m\,b^2\,a-2\,r^3\,m\,a-2\,r^3\,m\,b\right)\sqrt{1+m^2}\right)\left(1+m^2\right)\right)$$

```
> solve(%,m);
```

$$\frac{(a+b)\,r}{\sqrt{b^2\,a^2-r^2\,a^2-r^2\,b^2+r^4}},\ -\frac{(a+b)\,r}{\sqrt{b^2\,a^2-r^2\,a^2-r^2\,b^2+r^4}}$$

Figure 8: (a) Mechanical model of Regiomontanus' problem. (b) Expression for reaction in the slope constraint from Mechanical Expressions and its solution in Maple

Figure 8 shows the mechanical Expressions model used to analyse the problem where the earth is no longer flat. We have constrained the distances of the top and bottom of the "screen" from the centre of the each. We have also constrained the slope of the line OC. [S5] is an interactive online version of this model.
The reaction in the slope is obtained from Mechanical Expressions, and this time the use of Maple is justified in obtaining a solution (figure 8b). This solution may be pasted back into Mechanical Expressions (figure 9)

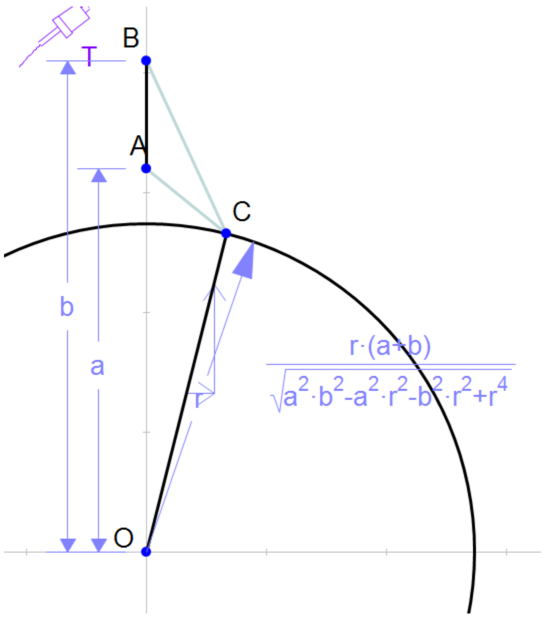


Figure 9: One equilibrium solution pasted back into Geometry Expressions

5. Minimum Perimeter Triangle

Another classic problem from [5] is to find the minimum perimeter triangle which can be inscribed in a given acute angled triangle.

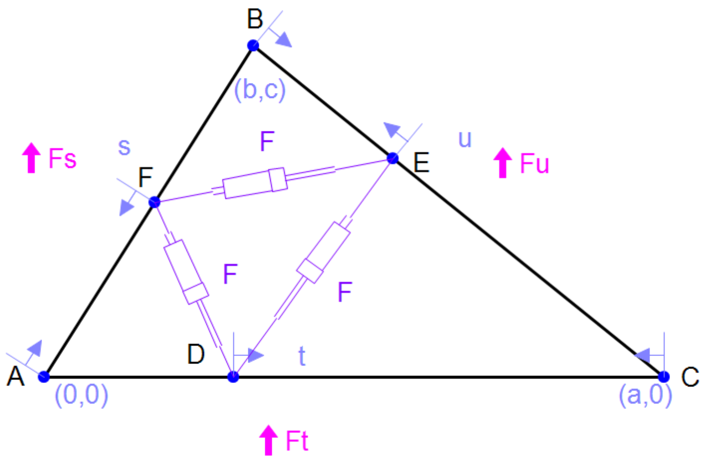


Figure 10: Mechanical Expressions model for the minimum perimeter inscribed triangle.

A Mechanical Expressions model for this problem (figure 10) puts constant force actuators along each side of the inscribed triangle. The potential energy of each actuator is force times distance. If each actuator has the same constant force, then the potential energy of the system is proportional to

the perimeter of the inscribed triangle. Hence minimizing potential energy is equivalent to minimizing the perimeter of the inscribed triangle [S6].
The vertices ABC of the original triangle are constrained by their coordinates. Without loss of generality, we place A at the origin and C on the x-axis. Vertices of the inscribed triangle DEF are constrained by their parametric locations on the lines CA, BC, AB. (The parametric location on a line segment is defined to be the proportion along the segment).
Values for the reaction force in each of these constraints may be computed in Mechanical Expressions. The equilibrium location can be found by solving this system of 3 equations (figure 11).

$$Fs := \frac{(2\,s\,c^2 - 2\,(-s\,b + (1-t)\,a)\,b)\,F}{2\sqrt{s^2\,c^2 + (-s\,b + (1-t)\,a)^2}} + \frac{(-2\,(-s\,b + u\,a + (1-u)\,b)\,b - 2\,(-s\,c + (1-u)\,c)\,c)\,F}{2\sqrt{(-s\,b + u\,a + (1-u)\,b)^2 + (-s\,c + (1-u)\,c)^2}}$$

$$Ft := -\frac{(-s\,b + (1-t)\,a)\,a\,F}{\sqrt{s^2\,c^2 + (-s\,b + (1-t)\,a)^2}} - \frac{(-u\,a + (1-t)\,a - (1-u)\,b)\,a\,F}{\sqrt{(1-u)^2\,c^2 + (-u\,a + (1-t)\,a - (1-u)\,b)^2}}$$

$$Fu := \frac{(-2\,(1-u)\,c^2 + 2\,(-u\,a + (1-t)\,a - (1-u)\,b)\,(-a + b))\,F}{2\sqrt{(1-u)^2\,c^2 + (-u\,a + (1-t)\,a - (1-u)\,b)^2}} + \frac{(2\,(-s\,b + u\,a + (1-u)\,b)\,(a-b) - 2\,(-s\,c + (1-u)\,c)\,c)\,F}{2\sqrt{(-s\,b + u\,a + (1-u)\,b)^2 + (-s\,c + (1-u)\,c)^2}}$$

```
> solve({Fs=0,Ft=0,Fu=0},{s,t,u});
```

$$\{u = u, s = s, t = -\frac{1 - u - s + s\,u}{-1 + u + s}\},\ \{s = \frac{a\,b}{b^2 + c^2}, t = \frac{a - b}{a}, u = \frac{c^2 - a\,b + b^2}{c^2 - 2\,a\,b + a^2 + b^2}\},$$

$$\{t = t, u = 1, s = 0\}$$

Figure 11: Reaction forces for the minimum inscribed triangle generated by Mechanical Expressions and solved for equilibrium in Maple

Copying the solutions back into Geometry Expressions (figure 12), we can observe that D E and F lie at the feet of the altitudes of the original triangle.

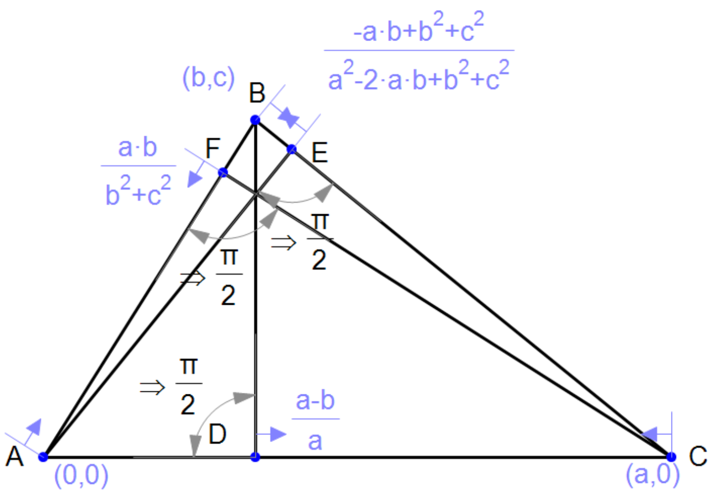


Figure 12: Geometric properties of the solution may be obtained by replacing the constraint values s,t,u with their equilibrium solutions.

6. Conclusion

Mechanical Expressions is designed to give symbolic answers to mechanical models. In this paper, however, we have shown how it can be used in conjunction with a CAS to solve geometric optimization problems. Mechanical Expressions has a number of advantages over traditional mechanics software in this context. The fact that it uses a symbolic Lagrangian than a numeric Hamiltonian formulation results in the acquisition of simple usable expressions which characterize equilibrium. Solving these can yield explicit representations of the geometric problem solutions. Mechanical Expressions is designed to allow the expression of simplified idealized models. Hence it is easy to express such idealized entities as a zero natural length spring or a constant force actuator.

The Mathematical Mechanic [1] supplies some general rules for formulating mechanics models whose minimum energy configuration corresponds to the solution of a given problem. Examples of these rules include the use of zero natural length springs for a least sum of squares problem and using constant force actuators for a minimum total length problem. Methods are also presented for deriving geometrical information from the solutions. These tend to be implicit, but can lead to very slick proofs of the results. The benefit of using the symbolic mechanics approach is that one uses the methods of the book to create a model, but solving the model is more routine and more explicit.

As a final comment, we would point out that the approach pursued in this paper can be seen as a cartoon rendering of quantum computing. In quantum computing, the problem to be solved is expressed as a quantum mechanical energy minimization, then solved by a (currently notional) quantum computer. In this paper a geometry problem is expressed as an energy minimization in Newtonian mechanics then solved by CAS.

References

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[3] Todd P. (2009) “Looking Forward to Interactive Symbolic Geometry” in Understanding Geometry for a Changing World, 71st Yearbook, NCTM, Editors: Tim Craine, Rheta Rubenstein, pp.349-365

[4] Kalman D.(2007) “Solving the Ladder Problem on the Back of an Envelope”, *Mathematics Magazine 80 (3)* 163-182

[5] Dorrie H. (1963) “100 Great Problems of Elementary Mathematics: their history and solutions”, Dover Publications, New York NY

[6] James T. & S. Jackson (2001) “Rugby and Mathematics: a Surprising Link among Geometry the Conics and Calculus”, Mathematics Teacher 94(8) 649-654

Software packages

[MechExpressions] Mechanical Expressions, a product of Saltire Software 2014
www.mechanicalexpressions.com

[GeomExpressions] Geometry Expressions, a product of Saltire Software 2006
www.geometryexpressions.com

Supplemental Electronic Materials

[S1] Fermat Toricelli simulation on Euclids Muse website http://euclidsmuse.com/app?id=864
[S2] Model of constrained double pendulum on Euclid's Muse website
http://euclidsmuse.com/app?id=500
[S3] Ladder Problem mechanical simulation on Euclid's Muse website
http://euclidsmuse.com/app?id=865
[S4] Drive in Movie Problem simulation on Euclid's Muse Website
http://euclidsmuse.com/app?id=867
[S5] Regiomontanus' Problem mechanical simulation on Euclid's Muse website
http://euclidsmuse.com/app?id=977
[S6] Minimum Perimeter Inscribed Triangle mechanical simulation on Euclid's Muse Website
http://euclidsmuse.com/app?id=976